Recitation 4

September 17

Review

Problem 1. **Every row** of A has a pivot, so *columns* of A span $\mathbb{R}^2$. **Not every row** of A^T has a pivot, so *columns* of A^T **don't span** $\mathbb{R}^4$.

Since **not every column** of A has a pivot, columns of A are linearly **dependent**. In A^T, **every column** has a pivot, so columns are linearly **independent**.

If $A^T x = b$ has at least one solution, it should have *exactly one* solution: **every column** of A^T has a pivot, so every variable is *basic*, so there are *no free variables*, so the solution is unique (provided that it exists).

Problem 2.

- Matrix $A = \begin{bmatrix} -1 & 3 & 4 \\ 3 & -4 & -2 \\ 0 & -3 & -6 \end{bmatrix}$ has two pivots. **Not every row** has a pivot, so A is **not onto**. **Not every column** has a pivot, so A is **not one-to-one**.

- Matrix $B = \begin{bmatrix} 4 & -1 & 3 \\ 4 & -4 & -2 \end{bmatrix}$ has two pivots. **Every row** has a pivot, so B is onto. **Not every column** has a pivot, so B is **not one-to-one**.

Problem 3. Transformation S sends a vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the vector $S(x) = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$ Therefore, if we take $cx = c\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} -cx_1 \\ -cx_2 \end{bmatrix}$ for any scalar $c \in \mathbb{R}$, and apply S, we will get $S(cx) = \begin{bmatrix} -cx_1 \\ -cx_2 \end{bmatrix} = c\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} = cS(x)$. So $S(cx) = cS(x)$ and the first property of linear transformations is satisfied.

Similarly for the second property. Take any two vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then $x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$, and $S(x+y) = \begin{bmatrix} -(x_1 + y_1) \\ -(x_2 + y_2) \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} -y_1 \\ -y_2 \end{bmatrix} = S(x) + S(y)$. So the second property of linear transformations is also satisfied, and so S is a linear transformation.

To find the matrix of S, first apply S to $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and get $S(e_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. This will be the first column of the matrix of S.

Apply S to $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and get $S(e_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. This is the second column of the matrix of S. So the resulting matrix is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Problem 4. Applying transformations T and S to the standard basis vectors e_1, e_2, e_3 we get the matrices of the transformations to be

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

To find the matrix of F, which is the **composition** of T and S, we need to simply multiply the two matrices:

$$F = ST = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Notice that the order of multiplication matters, and that F is actually equal to S.

Problem 5. $AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} = AC.$

The matrix A is **not invertible**, because its determinant is zero, $2 \cdot 6 - (-4) \cdot (-3) = 0$.

Problem 6. If X is invertible, the we can multiply both sides of the equation $XY = XZ$ by X^{-1}, and get $X^{-1}XY = X^{-1}XZ$, so $Y = Z$. So the "cancellation law" works fine if X is invertible.

Problem 7. If one knows A and AB, then $B = A^{-1} \cdot AB$ provided A is invertible. Here this is the case, and $A^{-1} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. Then

$$B = A^{-1} \cdot AB = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 8 & 20 \\ 5 & 3 & 8 \end{bmatrix}$$

Problem 8. Need to write augmented matrix $[C|I_3]$, do row reduction to get $[I_3|C^{-1}]$. Then

$$C^{-1} = \begin{bmatrix} -2 & -3 & 3 \\ 2 & 5 & -6 \\ -1 & -3 & 4 \end{bmatrix}$$

Problem 9. We can multiply both sides of the equation $C^{-1}(A + X)B^{-1} = ABC$ by C on the left and by B on the right to get $A + X = CABCB$, and so $X = CABCB - A$.

We never used that A was invertible, so it could have been any matrix at all.